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Effect of External Fluctuations on the Fréedericksz Transition in an Analogue Simulator

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The influence of multiplicative external fluctuations (noise) on the phenomenological equation describing the Fréedericksz transition has been studied by means of an electronic analogue simulator. Measurements were made of the stationary probability density for a wide range of fluctuation intensities and correlation times, for both dichotomous and Gaussianly distributed noise. For dichotomous forcing, the resultant phase diagrams at particular values of the field intensity parameter were found to be in satisfactory agreement with exact theoretical predictions by Horsthemke *et al.* In the (physically more realistic) case of Gaussian fluctuations, for which no theory is currently available, the results obtained were distinctively different. A physically motivated discussion is given to account for the interesting differences and similarities of behavior found for the two types of external noise.

KEY WORDS: Fréedericksz transition; analogue simulation; external noise; colored noise; dichotomous noise; noise-induced transitions.

1. INTRODUCTION

The macroscopic behavior of a nonlinear system can often be substantially modified by the effect of a fluctuating environment.[1] This is especially the case if the fluctuations (noise) are coupled to the system multiplicatively, when they can induce transition phenomena which cannot occur under deterministic environmental conditions. Although the phenomenological stochastic differential equations associated with such systems are difficult to solve,[1,2] progress can often be made by use of certain idealizations, such as regarding the stochastic variable to be white and Gaussianly distributed; the white noise idealization is a good approximation when the environmental fluctuations are rapidly varying compared with the characteristic time

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scale of the system. An appropriate Fokker–Planck equation may be obtained and its solutions found, at least for the stationary probability densities. However, the phases displayed by such systems are strongly dependent on the correlation time of the noise used to drive them.[3,4] It is thus desirable to relax the restriction to white noise and investigate the behavior of these systems as a function of noise correlation time τ_c. Although, as a consequence of the central limit theorem, the environmental fluctuations can still be considered Gaussian, no theory exists for which exact analytical results can be obtained in the case of a nonlinear system driven by a Gaussian, nonwhite, process with arbitrary τ_c.

A phenomenon of particular interest in this context is the *Fréedericksz transition* which occurs in nematic liquid crystals[5] and in the orbital texture[6] of dipole-locked superfluid ^{3}He-A. In each case, a thin slab of liquid held between parallel plates is subjected to a magnetic field such that the alignment of the liquid imposed by the plates lies perpendicular to that favored by the field. The texture remains unaffected by this competition until a critical value H_c of the field is reached, and the transition takes place. For fields above H_c, the texture is increasingly influenced by the field, the effect being strongest midway between the plates. Fluctuations of H about its average value are expected[3] to exert a marked influence on the critical value at which the transition occurs and on the phases displayed by the system.

To calculate the effect of field fluctuations on the Fréedericksz transition is, for the general case, very difficult. Close to the transition, however, where the alignment angle remains close to its zero-field value, the phenomenological equation can be approximated to a high degree of accuracy. Under these conditions, the resultant equation can be solved exactly[3] for two situations: for white, Gaussianly distributed noise; and for exponentially correlated, dichotomous noise. The (physically more realistic) case of colored Gaussian noise cannot be treated exactly; the hope, however, is that, at least qualitatively, the behavior of a given system will not be strongly dependent on the type of noise used to drive it. Recent work[7] by Kai *et al.* on the (closely related) electrohydrodynamic instability would seem in part to substantiate this hope. The latter authors showed that shifts in the threshold voltage V_c needed to cause a nematic liquid crystal to undergo a transition to the Williams domain were not strongly dependent on noise type.

In this paper, we describe how we have applied the technique of electronic analogue simulation[2] to the problem, in two separate ways. First, we

[2] For a review of electronic analogue simulation see ref. 8, especially the chapters by Fronzoni (Chapter 8) and McClintock and Moss (Chapter 9).

have studied the phenomenological equation under dichotomous, colored, external noise; the measurements are compared with exact theoretical predictions.[3] Second, we have used the same electronic simulator to investigate the effect on the system of colored, Gaussian noise, a situation for which, as already mentioned, no analytic theory exists. It was hoped in this way to gain some insight into the extent to which one can reliably infer the behavior of a system under Gaussian colored noise from studies of its response to dichotomous fluctuations.

2. ANALOGUE SIMULATION OF THE FRÉEDERICKSZ TRANSITION

It has been shown[3] that in the vicinity of the Fréedericksz transition, a nematic liquid crystal should evolve according to the approximate stochastic differential equation

$$\tau_0 \dot{\theta} = -\theta + (h + \eta_t)^2 (\theta - \tfrac{1}{2}\theta^3) \tag{1}$$

where $h = H/H_c$, $\eta_t = \eta_t/H_c$, H is the average magnetic field intensity, H_c is the deterministic critical field (i.e., in the absence of fluctuations) at which the transition just occurs, and τ_0 is the zero-field relaxation time. The state variable θ is the angle between the nematic director and the preferred direction of molecular alignment in the absence of a magnetic field. Equation (1) is used as the basis of the analogue simulation, with the stochastic field fluctuations η_t being modeled as *either* dichotomous *or* Gaussian noise processes.

A schematic diagram of the electronic circuit used to mimic Eq. (1) is shown in Fig. 1. The voltage summation, integration, and amplification operations were accomplished by standard operational amplifiers, while

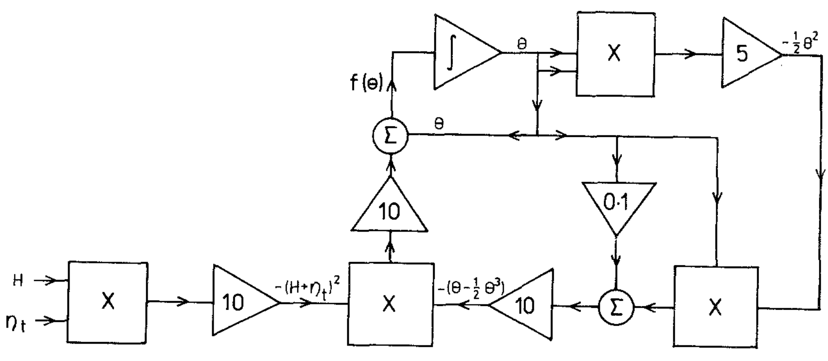

Fig. 1. Block diagram of the analogue electronic used to model Eq. (1).

voltage multiplication was performed by commercially available integrated circuit multipliers (Analog Devices, type AD534KD). The multipliers also have the effect of dividing their outputs by ten, hence the need for amplification by an appropriate factor at various points in the circuit. Using the above operations, a function $f(\theta)$ was constructed, where

$$f(\theta) = +\theta - (h+\eta_t)^2 \,(\theta - \tfrac{1}{2}\theta^3) \tag{2}$$

$f(\theta)$ was then passed to an integrator with a transfer function,

$$\theta = \frac{-1}{\tau_0} \int f(\theta)\, dt \tag{3}$$

thus reproducing Eq. (1). The voltage waveform $\theta(t)$ characterizing the state of the system, available at the output of the integrator, was analyzed by means of a digital data processor (Nicolet LAB80) which constructed the stationary probability density $P_{st}(\theta)$. It was thus possible, by changing the appropriate parameters, to map out the transitional behavior of the stationary probability density as a function of the correlation time and intensity of the external noise.

To obtain results comparable with theory,[3] η_t was initially modeled as dichotomous noise. The dichotomous noise generator constructed for this purpose is shown schematically in Fig. 2. This instrument was built around a single bistable device incorporating an operational amplifier. The bistable device was switched between its two stable states Δ and $-\Delta$ when the input voltage attained magnitudes greater than a threshold voltage set

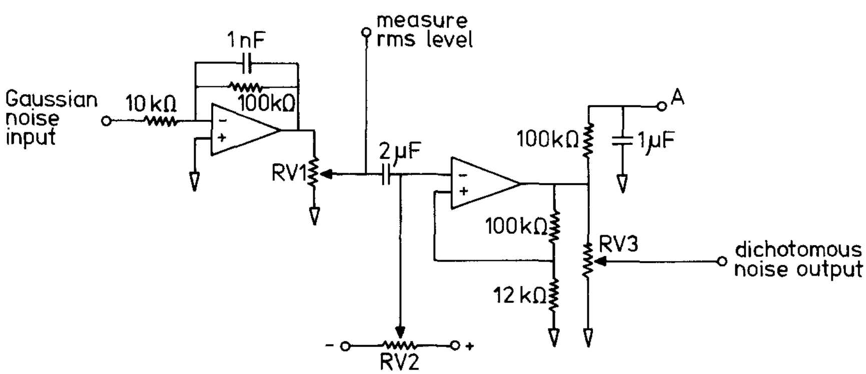

Fig. 2. Circuit of the dichotomous noise generator used to drive the analogue simulator of Fig. 1. The right-hand operational amplifier acts as a bistable device which switches between its stable states whenever its input exceeds a positive or negative threshold voltage, thereby converting the input Gaussian noise into an exponentially correlated dichotomous output voltage.

internally in the device. Thus, by applying pseudo-white Gaussian noise to the input and by varying its intensity using RV1, the number of transitions between the two states and hence the correlation time of the dichotomous noise could be altered. Using the same data processor, the correlation time of the noise was determined as a funtion of the noise intensity measured at the input of the bistable, and a calibration curve constructed. A desired correlation time could then be selected by setting the input noise intensity to the appropriate value. The dichotomous noise was found to be exponentially correlated with a correlation function

$$C(\tau) = \Delta^2 e^{-|\tau|/\tau_c}$$

where τ_c can be regarded as a measure of the average waiting time in one of the stable states. Offsets in the noise generator were compensated for by adjustment of RV2 such that the potential at A was zero, thereby ensuring that the mean output was zero. The output intensity of the noise generator could be controlled using the resistive attenuator RV3.

The Gaussian noise was obtained from a Wandel & Goltermann model RG1 noise generator, which produces an accurately Gaussian noise voltage of constant spectral intensity over the band range 0–100 kHz. The noise source was used to drive the dichotomous noise generator, as well as for the Gaussian noise experiments themselves. For the latter experiments the noise was initially passed through the active, low-pass filter shown schematically in Fig. 3. This ensured that the noise had a well defined correlation time τ_c and a correlation function

$$C(\tau) = \sigma^2 e^{-|\tau|/\tau_c}$$

The variance of the noise σ^2 was measured at the output of the filter by use of a specialized integrated circuit (Analog Devices type AD536AKD). The correlation time of the noise is determined by the time

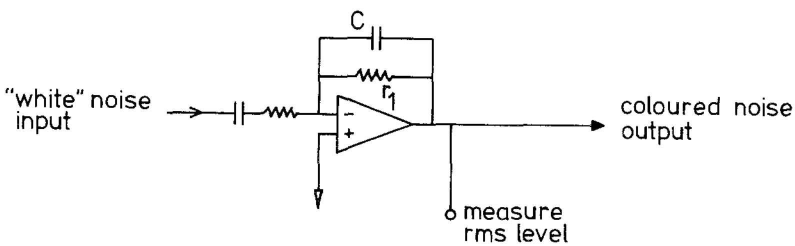

Fig. 3. Simple active low-pass filter used to define the correlation time of the Gaussian noise used to drive the analogue simulator of Fig. 1.

constant $r_1 C$: Thus, by changing C it was possible to alter the correlation time of the noise independently of the gain of the filter and hence the noise intensity.

3. RESULTS OF THE ANALOGUE EXPERIMENTS

Horsthemke *et al.*[3] presented the results of their calculations in the form of phase diagrams mapped out on the δ–$\gamma\tau_0$ plane. Here δ is the reduced noise intensity ($=\Delta/H_c$ or σ/H_c, depending on the type of noise used) and γ is the inverse correlation time of the noise. Phase diagrams were presented for three fixed values of the magnetic field intensity, $H=0.99H_c$, $1.01H_c$, and $1.3H_c$. The dichotomous and Gaussian noise experiments were carried out in essentially the same manner. After setting the magnetic field intensity parameter to one of the three values of interest,

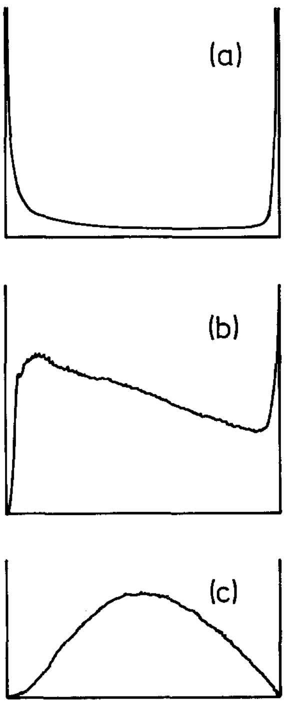

Fig. 4. Typical examples of stationary densities measured for the analogue simulator of Fig. 1 when driven by exponentially correlated dichotomous noise with $h=1.01$, $\delta=0.5$, and (a) $\gamma\tau_0=1.15$, (b) $\gamma\tau_0=3.4$, (c) $\gamma\tau_0=6.8$. The vertical lines indicate the positions of the supports.

the noise intensity was then fixed and the transitional behavior of $P_{st}(\theta)$ recorded for different values of the noise correlation time. The noise intensity was then changed and the procedure repeated. This was continued until the parameter space of interest had been covered. The criterion used to determine whether or not a phase transition had occurred was the same as that laid down by Horsthemke *et al.*, who argued that qualitative changes in the extrema of $P_{st}(\theta)$ were what distinguished the different phase regions. To aid in the recording of the phase diagrams we thus assigned a unique symbol to each of the phases exhibiting qualitatively different extrema; the symbols were then plotted directly onto the δ–$\gamma\tau_0$ diagram as the corresponding probability densities were observed.

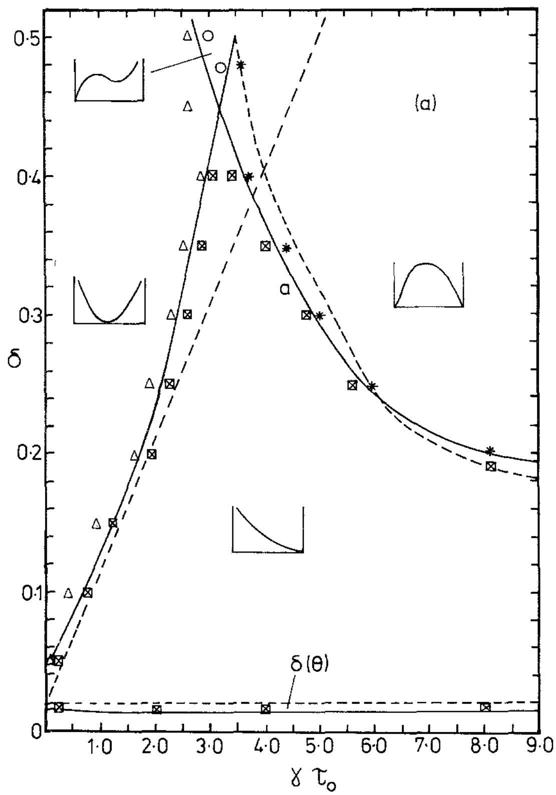

Fig. 5. Phase diagrams measured for the analogue simulator of Fig. 1 when driven by exponentially correlated dichotomous noise with (a) $h = 0.99$, (b) $h = 1.01$, (c) $h = 1.30$. The stationary densities measured in different parts of the δ–$\gamma\tau_0$ plane were as indicated by the inset sketches. The solid curves are guides to the eye, to indicate the experimental transitional boundaries, and the dashed curves represent theoretical predictions by Horsthemke *et al.*[3]

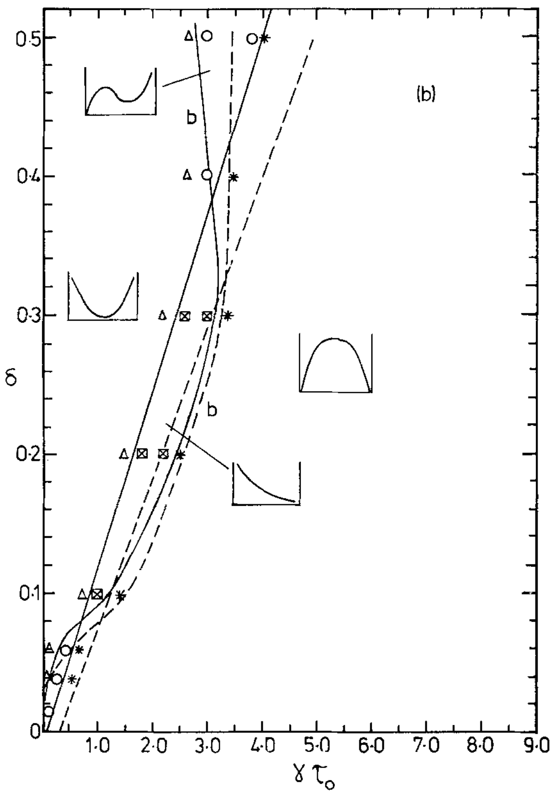

Fig. 5 (*continued*)

Some examples of stationary densities recorded for dichotomous noise are shown in Fig. 4, for the parameter values indicated. Using a large number of densities such as these, the phase diagrams shown by the solid curves of Fig. 5 were then constructed. It may be noted, first, that all of the phase regions predicted[3] by Horsthemke *et al.* (dashed curves) were in fact observed. Second, however, it is evident that the precise positions of the phase boundaries differ slightly from those predicted. We do not regard these discrepancies as significant: partly because of the systematic errors and nonidealities, amounting typically to a few percent, that are to be expected in analogue simulators[8]; and partly because the transitional behavior of $P_{st}(\theta)$ is not absolutely sharply defined, allowing a small amount of room for interpretation as to whether or not a phase boundary has been crossed. Subject to these qualifications, we regard the agreement between the theoretical predictions and the analogue experiments as gratifying.

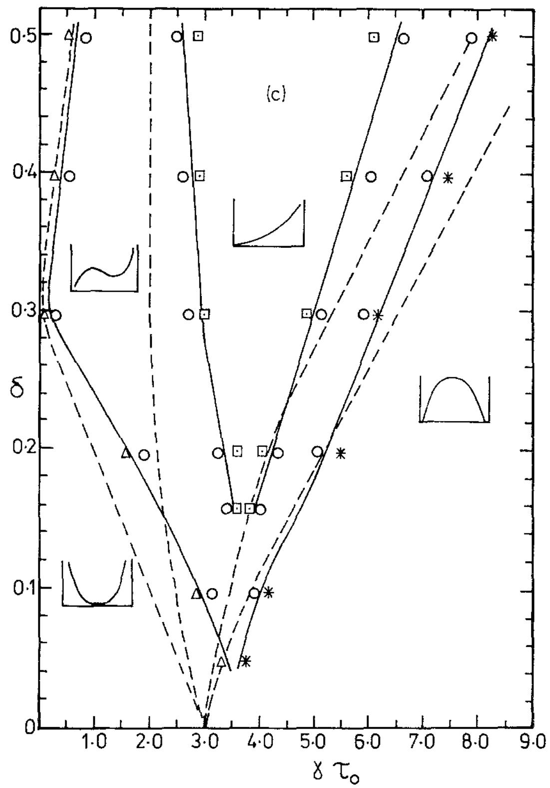

Fig. 5 (*continued*)

The experimental results obtained for Gaussian noise are shown in Figs. 6 and 7. It can be seen immediately that there are some significant differences from the equivalent dichotomous noise results, both in the positioning and shape of the transition boundaries and, more importantly, in the type of phases observed. First, consider the two phase diagrams for which $H = 0.99H_c$ (Figs. 5a and 7a). Only two of the predicted phases for dichotomous noise were observed in the Gaussian noise case, and then the associated phase regions occupied different areas in parameter space. Furthermore, phases which were not predicted for dichotomous noise were observed for the Gaussian, the converse also being true. Similar results are apparent for the other two sets of phase diagrams (comparisons of Figs. 5b with 7b, and Figs. 5c with 7c). In general, predicted phases, characterized by an *increasing* probability density at the upper support were *not* observed for Gaussian noise. This is most apparent in the two phase diagrams (Figs. 5c and 7c) for which $H = 1.3H_c$. Here, three of the predicted phases

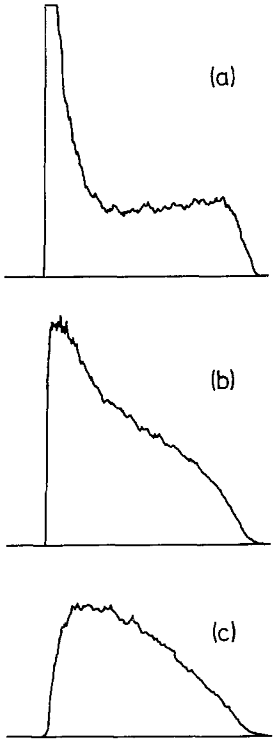

Fig. 6. Typical stationary densities measured for the analogue simulator of Fig. 1 when driven by exponentially correlated Gaussian noise with $h = 1.01$, $\delta = 0.5$, and (a) $\gamma\tau_0 = 0.33$, (b) $\gamma\tau_0 = 1.7$, (c) $\gamma\tau_0 = 2.7$. The top of the density in (a) has been truncated by the graph plotting system.

show an increase in the probability densities at the upper support, whereas none of these are observed in the Gaussian noise phase diagrams.

4. DISCUSSION

It is evident from the results of the preceding section that the rich transitional behavior predicted for dichotomous noise is not observed in the case of Gaussian noise. We believe this is simply a consequence of the limited state space of the dichotomous noise process. Due to the overdamped nature of the system, the upper state of the dichotomous processes imposes an upper limit on the state space of the system. Thus, on average, the system has time to relax to this upper state when the correlation time of the noise becomes comparable with the relaxation time of the system. This results in a "buildup of probability" at this upper state limit, i.e., at the upper support of the probability density. Such behavior would not be expected to occur if the noise were allowed to have a more general state space, as is the case with Gaussian noise.

An important similarity between the Gaussian and dichotomous noise results can now be explained. In the limit of rapidly varying fluctuations, the system displays the same phase for both types of noise. This is due to the fact that the system no longer has time to relax to any state limit imposed *by the noise*, i.e., the phase of the system is essentially independent of the state space of the noise. Thus, it is only to be expected that, in the

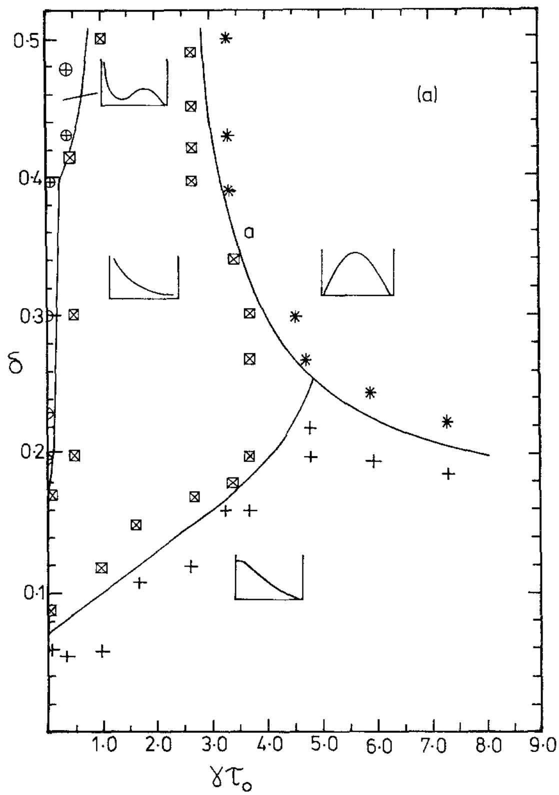

Fig. 7. Phase diagrams measured for the analogue simulator of Fig. 1 when driven by exponentially correlated Gaussian noise with (a) $h = 0.99$, (b) $h = 1.01$, (c) $h = 1.30$. The stationary densities measured in different regions of the δ–$\gamma\tau_0$ plane were as indicated by the inset sketches. The solid curves are guides to the eye to indicate the experimental transitional boundaries.

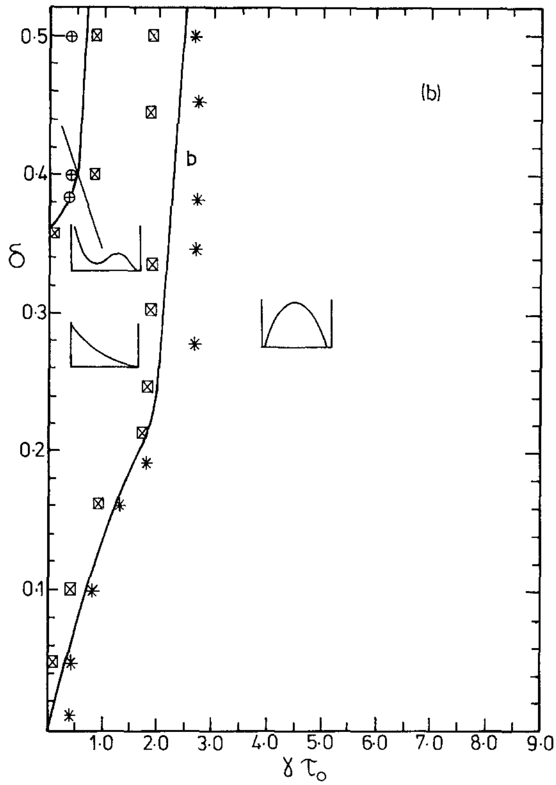

Fig. 7 (*continued*)

white noise limit, the qualitative behavior of a system will be independent of the type of noise used to drive it.

Another important similarity in the results can be seen by comparing the two sets of phase diagrams (Fig. 5a with 7a, and Fig. 5b with 7b) for which $H = 0.99H_c$ and $H = 1.01H_c$. The transition boundaries labeled *a* and *b* represent the Fréedericksz transition in the presence of noise. The noise has had the effect of shifting the mean critical field H_c required to induce the transition. The positioning of these boundaries is similar in both the dichotomous and Gaussian noise phase diagrams; the shapes of the boundaries are also similar. Thus, it would seem that here, too, the shifts in H_c induced by the noise are independent of the type of noise used to induce the transition. The Fréedericksz transition is a hard transition[3] involving the lower support of the probability density. This is in contrast to the other hard transitions predicted for dichotomous noise, which involve the upper support of the probability density; as already discussed, these were not

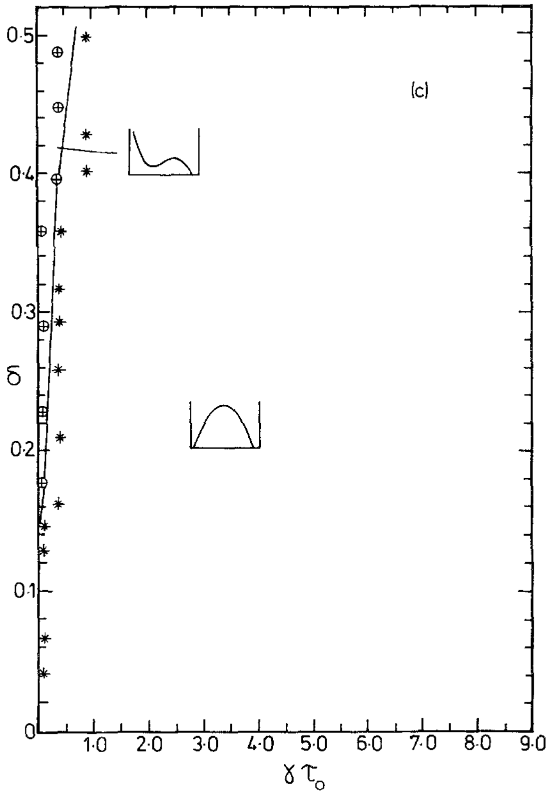

Fig. 7 (*continued*)

observed for Gaussian noise. For values of $H < H_c$ there exists a single deterministic state in which the system resides. It is this single state which imposes a lower state limit on the system in the presence of noise, the Fréedericksz transition being a consequence of this lower state limit. This is a physical limit imposed by the system and is not a consequence of the state space of the dichotomous noise. Thus, one may expect, qualitatively at least, that the shifts in H_c will not be dependent on the type of noise being applied.

5. CONCLUSIONS

We have confirmed exact analytical results, obtained by Horsthemke *et al.*,[3] for the effect of dichotomous field fluctuations on the Fréedericksz transition. It is, however, apparent that certain aspects of these results cannot be generalized to describe the behavior of the system under the influence of other types of noise. The limited state space of the dichotomous noise imposed artificial limits on the state space of the system; the resulting hard transitions were thus a characteristic of the dichotomous noise alone. Qualitatively similar results for the effect of dichotomous and Gaussian noise on the mean critical field H_c were obtained. This can be attributed to the fact that the Fréedericksz transition is a consequence of a physical, lower state limit imposed by the system itself, and hence is largely independent of noise type. It was also shown that in the pseudo-white noise limit of very rapidly varying fluctuations the behavior of the system was again independent of noise type. However, from the theorist's point of view, the prime motivation for modeling the field fluctuations as a dichotomous noise process was to obtain results for an arbitrary correlation time. We have shown that it is just in this strongly colored limit that the state space of the noise becomes important and, although useful information can be obtained, caution must be exercised when generalizing the results to other noise types.

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